Int. J. Solids Structures Vol. 26, No. 2, pp. 237–249, 1990
Printed in Great Britain.

0020-7683/90 $3.00 + .00

THE GREEN'S FUNCTIONS OF CLAMPED SEMI-INFINITE VIBRATING BEAMS AND PLATES

NORBERT ORTNER and PETER WAGNER†
Institut für Mathematik und Geometrie, Universität Innsbruck, Technikerstr. 13, A-6020 Innsbruck, Austria

(*Received* 9 *March* 1989)

Abstract—The Green's functions of semi-infinite, transversally vibrating homogeneous beams and isotropic resp. special orthotropic plates, which are clamped along the boundary, are represented by definite integrals. Some of these are evaluated by numerical procedures.

1. INTRODUCTION

In the classical books of Todhunter and Pearson (1893) and of Girkmann (1963), the solutions of several initial-boundary value problems for semi-infinite bars and plates are represented by formulae containing tabulated functions. The boundary conditions prescribed in most of these problems correspond to freely supported boundaries. If the transversal deflection is denoted by w, then this amounts to prescribing the values of w and of $\partial^2 w/\partial x^2$ along the straight border $x = 0$. Since the derivation order in the two boundary conditions differs by 2, these problems are completely reducible to boundary value problems for second-order differential equations. Under these circumstances also the method of images applies.

However, the classical Dirichlet problem for fourth-order equations with respect to the space variables demands, as given data, the values of w and of $\partial w/\partial x$ along the border $x = 0$. Physically, this corresponds to a clamped boundary. Whereas the static cases of semi-infinite clamped bars and plates were solved explicitly (with the exception of clamped embedded plates), the dynamical versions of these problems remained open.

The initial-boundary value problem (mathematically: mixed Cauchy–Dirichlet problem) of a semi-infinite clamped bar has already been solved by determining the Green's function (Ortner, 1978). In the present paper, we intend to give explicit formulae for the Green's functions of the following semi-infinite clamped vibrating bars, resp. plates:

(i) elastically supported bar: Section 2.3,
(ii) special orthotropic plate: Section 3.2,
(iii) isotropic plate: Sections 4.2, 4.3.

As a corollary to the construction of the Green's function in (iii), we also deduce the Green's function of the semi-infinite clamped elastically supported static isotropic plate in Section 4.4. Furthermore, in Section 2.5, we give a detailed numerical analysis of the integral which represents the Green's function corresponding to (i) in the special case when there is no elastic bedding. Also, the definite integral which corresponds to (ii) is evaluated by numerical procedures.

The basic mathematical tools are the classical integral transformations, i.e. the Laplace transform with respect to the time variable and the Fourier transform with respect to one of the space variables. Moreover, we apply the theory of asymptotic expansions to the evaluation of certain definite integrals. The mathematical treatment of the existence and

† The authors are indebted to Dipl. Ing. Arthur Wagner for programming the numerical evaluation of the oscillating improper integral given in Section 3.2, which represents the Green's function of the clamped orthotropic plate.

the uniqueness of the Green's function of these (mixed) initial-boundary value problems has been left out of consideration.

2. THE TRANSVERSE VIBRATIONS OF AN ELASTICALLY SUPPORTED, HOMOGENEOUS BEAM WHICH IS CLAMPED AT ONE END AND INFINITELY LONG IN ONE DIRECTION

2.1. *Statement of the problem*

The displacements $w(x, t)$ along a beam (of constant cross-section) which is supported by an elastic medium and loaded along its principal axis satisfy the differential equation (Parkus, 1966, Chapter XI)

$$\frac{\partial^4 w}{\partial x^4} + \frac{1}{c^2}\cdot\frac{\partial^2 w}{\partial t^2} + k^2 w = \frac{q}{EJ}. \tag{1}$$

Here, $c^2 = EJ/\rho F$, $E =$ Young's modulus, $J =$ moment of inertia of the cross-section, $\rho =$ density, $F =$ area of the cross-section, $k^2 = b/EJ$, $b =$ bedding coefficient = modulus of foundation, $q =$ external load.

We consider a semi-infinite beam, which is clamped at the end $x = 0$, i.e $w(0, t) = (\partial w/\partial x)(0, t) = 0$, and of infinite extension in the positive x-direction. We assume that $w(x, t) = 0$ for $t < 0$. The Green's function (also called the influence function or singularity function) $G_\xi(x, t)$, $\xi > 0$, represents the displacements along this beam produced by a concentrated instantaneous force at the point $x = \xi$ at the time $t = 0$, which amounts to putting $q(x, t) = EJ\delta(x-\xi)\otimes\delta(t)$, δ being Dirac's function.

2.2. *Application of the Laplace transform*

The Laplace transform g_ξ of G_ξ with respect to t is given by

$$g_\xi(x,p) = \mathscr{L}G_\xi(x,t) = \int_0^\infty e^{-pt}\, G_\xi(x,t)\, dt.$$

Then g_ξ is the Green's function of the ordinary differential operator $(d^4/dx^4)+(p^2/c^2)+k^2$ in the interval $(0, \infty)$. The function g_ξ fulfills:

$$g_\xi^{(iv)} + \left(\frac{p^2}{c^2} + k^2\right) g_\xi = 0, \quad 0 < x < \infty, \quad x \neq \xi, \quad g_\xi(0) = g'_\xi(0) = 0;$$

furthermore, g_ξ has to be bounded and twice continuously differentiable in $x = \xi$ and:

$$\lim_{\varepsilon\to 0}\,[g_\xi'''(\xi+\varepsilon) - g_\xi'''(\xi-\varepsilon)] = 1.$$

By these requirements, g_ξ is uniquely determined, and we obtain:

$$g_\xi = \frac{1}{8\alpha^3}\{e^{-\alpha|x-\xi|}\,[\cos\alpha(x-\xi) + \sin\alpha|x-\xi|] + e^{-\alpha(x+\xi)}\,[\cos\alpha(x+\xi) - \sin\alpha(x+\xi) - 2\cos\alpha(x-\xi)]\},$$

where for abbreviation: $4\alpha^4 = (p^2/c^2)+k^2$.

2.3. *Inversion of the Laplace transform*

In order to invert the Laplace transform, we make use of the formula

$$\mathscr{L}^{-1}\left[\left(\frac{p^2}{c^2}+k^2\right)^{-1/2}\phi\left(\sqrt{\frac{p^2}{c^2}+k^2}\right)\right]=c\int_0^{ct}\mathscr{L}^{-1}\phi(\tau)J_0(k\sqrt{c^2t^2-\tau^2})\,\mathrm{d}\tau$$

(cf. Colombo and Lavoine, 1972, p. 93). Since

$$\mathscr{L}^{-1}\left(\frac{1}{\sqrt{p}}\mathrm{e}^{-z\sqrt{p}}\right)=\frac{Y(t)}{\sqrt{\pi t}}\mathrm{e}^{-z^2/4t},\quad \mathrm{Re}\, z\geqslant 0$$

[see Oberhettinger and Badii, 1973, p. 258, formula (5.87); $Y(t)$ denotes the Heaviside function], we are able to compute $\mathscr{L}^{-1}g_\xi$ by setting $z=((1\pm i)/\sqrt{2})|x\pm\xi|$ and separating the real and imaginary parts. This yields:

$$G_\xi(x,t)=\frac{cY(t)}{2\sqrt{\pi}}\int_0^{ct}J_0(k\sqrt{c^2t^2-\tau^2})\cdot\left\{\sin\left[\frac{(x-\xi)^2}{4\tau}+\frac{\pi}{4}\right]-\sin\left[\frac{(x+\xi)^2}{4\tau}-\frac{\pi}{4}\right]\right.$$
$$\left.-\sqrt{2}\exp\left(-\frac{x\xi}{2\tau}\right)\cos\left[\frac{x^2-\xi^2}{4\tau}\right]\right\}\frac{\mathrm{d}\tau}{\sqrt{\tau}}.$$

2.4. *The fundamental solution*

From the above formula for G_ξ, we can easily deduce a fundamental solution E of the operator

$$P(\partial)=\frac{\partial^4}{\partial x^4}+\frac{1}{c^2}\cdot\frac{\partial^2}{\partial t^2}+k^2,\quad \partial=(\partial_x,\partial_t)=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right).$$

Physically, E describes the displacements of an infinite beam, supported by an elastic medium and acted upon by a concentrated, instantaneous force, i.e. $P(\partial)E=\delta=\delta(x)\otimes\delta(t)$. We can calculate E from G_ξ by shifting the boundary to infinity:

$$E(x,t)=\lim_{\xi\to\infty}G_\xi(x+\xi,t)=\frac{cY(t)}{2\sqrt{\pi}}\int_0^{ct}J_0(k\sqrt{c^2t^2-\tau^2})\sin\left(\frac{x^2}{4\tau}+\frac{\pi}{4}\right)\frac{\mathrm{d}\tau}{\sqrt{\tau}}.$$

(To obtain this equation, the Riemann–Lebesgue Lemma has to be applied.) E is unique as a fundamental solution of the quasi-hyperbolic operator $P(\partial)$ if the relatively weak condition

$$\exists\alpha>0:\mathrm{e}^{-\alpha t}E\in\mathscr{S}'$$

is imposed. Here $\mathscr{S}'$ denotes the space of temperate distributions, cf. Schwartz (1966). For a more extensive treatment of existence, uniqueness and construction of fundamental solutions, refer to Ortner (1980, 1987).

2.5. *The limiting case* k = 0

In the case where $k=0$, we have to substitute the factor $J_0(k\sqrt{c^2t^2-\tau^2})$ in the formula of Section 2.3 by the constant 1. The resulting integral was given for the first time by Ortner (1978). In this paper we discuss also, in detail, the representation of the solution of eqn (1) in 2.1 [for $k=0$ and subjected to the boundary conditions $w(0,t)=l(t)$, $(\partial w/\partial x)(0,t)=m(t)$, and the initial data which are given by $w(x,0)=g(x)$ and $(\partial w/\partial t)(x,0)=h(x)$] by means of the Green's function G_ξ. Note that the computation of G_ξ given in Ortner (1978) depends on the evaluation of the definite integral

$$\int_0^\infty f(ax)f(bx)\cos x^2\,\mathrm{d}x,\quad f(x)=\sin x-\cos x+\mathrm{e}^{-x},$$

which can alternatively be found in Ortner (1982).

Next, let us remark on the numerical evaluation of G_ξ in the special case where $k = 0$. Putting the dimensionless variables $u = x/\xi$, $s = ct/\xi^2$, $\sigma = ct/\tau$, $g(u,s) = G_\xi/c\xi$ we obtain from section 2.3:

$$g=\frac{\sqrt{s}}{2\sqrt{\pi}}\int_1^\infty\left\{\sin\left[\frac{(u-1)^2\sigma}{4s}+\frac{\pi}{4}\right]-\sin\left[\frac{(u+1)^2\sigma}{4s}-\frac{\pi}{4}\right]\right.$$
$$\left.-\sqrt{2}\exp\left(-\frac{u\sigma}{2s}\right)\cos\left[\frac{(u^2-1)\sigma}{4s}\right]\right\}\frac{d\sigma}{\sigma^{3/2}}.$$

Repeated partial integration yields the following asymptotic expansion (cf. Erdélyi, 1956):

$$g\sim\frac{\sqrt{s}}{2\sqrt{\pi}}\sum_{n=0}^{N}\frac{(2n+1)!!}{2^n}\left\{\mathrm{Im}\left[\frac{\mathrm{e}^{i(a-\pi/4)}}{(ia)^{n+1}}-\frac{\mathrm{e}^{i(b+\pi/4)}}{(ib)^{n+1}}\right]+\sqrt{2}\,\mathrm{Re}\left(\frac{e^d}{d^{n+1}}\right)\right\},$$

where $a = (u+1)^2/4s$, $b = (u-1)^2/4s$, $d = i(u+i)^2/4s$, which is valid for a, b, d being large. It is recommended that this expansion is broken before N attains the order of a, b, $|d|$, respectively. In fact, by Stirling's formula, we obtain

$$\frac{(2n+1)!!}{2^n\alpha^{n+1}}=\frac{2\Gamma(n+\frac{3}{2})}{\sqrt{\pi}\alpha^{n+1}}\approx 2^{3/2}\sqrt{e}\left(\frac{n+\frac{1}{2}}{\alpha e}\right)^{n+1},$$

and the last expression reaches its minimum approximately at $n = \alpha$ for a large value of α, say greater than 10.

For small values of a, b and d, g is best computed by using its power series expansion in a, b, d. For physical reasons, $\lim_{t\to\infty} G_\xi(x,t) = 0$ holds. From this, we infer that g may be written in the equivalent form

$$g=\frac{\sqrt{s}}{2\sqrt{\pi}}\int_0^1\left[\sin\left(a\sigma-\frac{\pi}{4}\right)-\sin\left(b\sigma+\frac{\pi}{4}\right)+\sqrt{2}\,\mathrm{Re}\,(e^{d\sigma})\right]\frac{\mathrm{d}\sigma}{\sigma^{3/2}}.$$

(This latter formula can also be deduced rigorously by showing that the above integral, extended from 0 to ∞, vanishes.) Upon expanding the integrand in a power series in σ, we obtain:

$$g=\frac{\sqrt{s}}{2\sqrt{2\pi}}\sum_{n=2}^{\infty}\frac{1}{n!(n-\frac{1}{2})}\{(-1)^{[(n-1)/2]}a^n-(-1)^{[n/2]}b^n+2\,\mathrm{Re}\,d^n\}.$$

Here, $[\alpha]$ stands for the largest integer $\leqslant\alpha$. This formula is adapted for numerical evaluation if the values of a, b and d do not exceed say 30 and if one uses at least 15 relevant digits in numerical processing.

Finally, let us consider the case of a small value of b, i.e. x near ξ, combined with large values of a and d, which means small times. Then the terms of the integral containing a and d are treated with the aid of an asymptotic expansion as above, whereas the term containing b is transformed into a power series using the formula

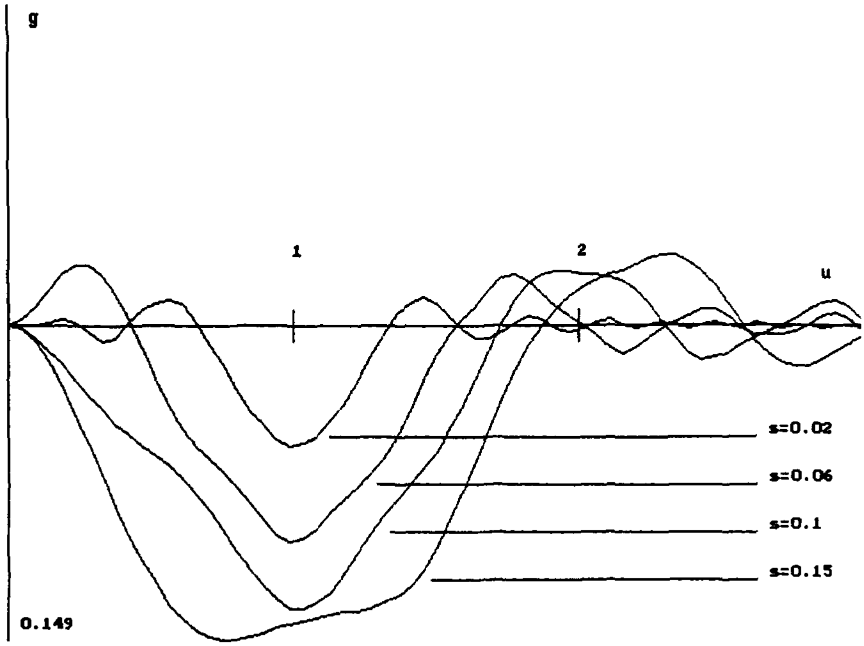

Fig. 1. Green's function of a semi-infinite one-sided clamped beam: initial phase of the wave.

$$\int_1^\infty \sin\left(b\sigma+\frac{\pi}{4}\right)\frac{d\sigma}{\sigma^{3/2}} = \sqrt{2} - \int_0^1 \left[\sin\left(b\sigma+\frac{\pi}{4}\right)-\frac{1}{\sqrt{2}}\right]\frac{d\sigma}{\sigma^{3/2}}.$$

In order to convey an idea of the shape of the beam during its vibration, we have calculated, utilizing the procedure outlined above, some deflection curves, which show the rapid oscillations in the beginning of the movement as well as the more regular fading-out wave later on (Figs 1, 2). Furthermore, we have determined the point and the time of maximal displacement: $u = 1.61$, $s = 0.52$, $g_{max} = 0.256$.

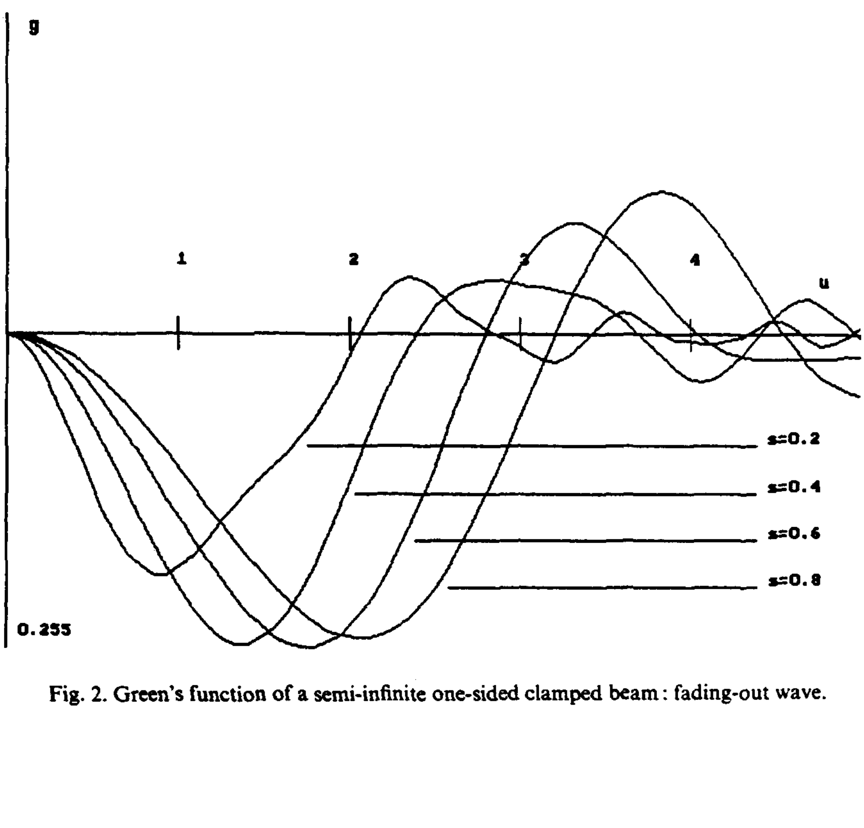

Fig. 2. Green's function of a semi-infinite one-sided clamped beam: fading-out wave.

3. THE TRANSVERSE VIBRATIONS OF A SEMI-INFINITE, ORTHOTROPIC PLATE CLAMPED ALONG ITS BOUNDARY

3.1. *Statement of the problem*

Let us now examine the vertical displacements $w(x, y, t)$ over a thin special orthotropic plate due to the load $p(x, y, t)$. w fulfills the differential equation

$$\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial y^4}+\frac{1}{c^2}\cdot\frac{\partial^2 w}{\partial t^2}=\frac{p}{K}. \tag{2}$$

Here $c^2 = K/c\rho$, h = thickness, ρ = density, $K = Eh^3/12(1-\nu^2)$, E = Young's modulus, ν = Poisson's ratio (cf. Nowacki, 1974, p. 348, eqn (16)]. We shall consider a semi-infinite plate, which is to occupy the region $-\infty < y < \infty$, $0 \leqslant x < \infty$ and is clamped along its boundary $x = 0$, i.e. $w(0, y, t) = (\partial w/\partial x)(0, y, t) = 0$. We proceed to determine the Green's function $H_\xi(x, y, t)$, $\xi > 0$, which, in analogy with Section 2, solves eqn (2) with right-hand side $p/K = \delta(x-\xi)\otimes\delta(y)\otimes\delta(t)$ and satisfies the above boundary conditions.

3.2. *Integral formula representing* H_ξ

Applying the Fourier transformation $\int w(x, y, t)\,e^{-iy\eta}\,dy$ with respect to y, eqn (2) assumes the form of the eqn (1) with $k = \eta^2$. Denoting by $G_\xi(x, y; k)$ the Green's function determined in the previous section we conclude, by inverting the Fourier transform, that

$$H_\xi(x, y, t)=\frac{1}{\pi}\int_0^\infty G_\xi(x, t; \eta^2)\cos(y\eta)\,d\eta.$$

On account of the formula (Oberhettinger, 1957, p. 71)

$$\int_0^\infty J_0(\eta^2 a)\cos(y\eta)\,d\eta=\frac{\pi|y|}{8a}\left(J_{-1/4}\left(\frac{y^2}{8a}\right)^2-J_{1/4}\left(\frac{y^2}{8a}\right)^2\right),\quad a>0,$$

we eventually find:

$$H_\xi(x, y, t)=\frac{c|y|Y(t)}{16\sqrt{\pi}}\int_0^{ct}\left\{J_{-1/4}\left(\frac{y^2}{8\sqrt{c^2t^2-\tau^2}}\right)^2-J_{1/4}\left(\frac{y^2}{8\sqrt{c^2t^2-\tau^2}}\right)^2\right\}$$

$$\cdot\left\{\sin\left[\frac{(x-\xi)^2}{4\tau}+\frac{\pi}{4}\right]-\sin\left[\frac{(x+\xi)^2}{4\tau}-\frac{\pi}{4}\right]-\sqrt{2}\,e^{-(x\xi/2\tau)}\cos\left[\frac{x^2-\xi^2}{4\tau}\right]\right\}$$

$$\times\frac{d\tau}{\sqrt{\tau}\sqrt{c^2t^2-\tau^2}}.$$

This formula has been evaluated with the aid of Simpson's rule. The individual integration stepsizes are adjusted to full wavelengths of the respective "main terms" of the integrand. If integration starts from a favourable intermediate τ-value between 0 and ct towards 0, the oscillation frequency of the three trigonometric terms increases indefinitely, whereas in the positive τ-direction, there is an indefinite increase of the oscillation frequency of the Bessel terms. These predominantly oscillating terms are denoted as "main terms" and their multiplying factors are called "side terms". Intermediate zeros of the side terms additionally limit the integration step sizes on account of improved preciseness. In the case where the arguments of the trigonometric terms weakly depend on τ (e.g. $x \to \xi$), the number of the subdivisions for the integration towards 0 has to be increased. Three-dimensional pictures

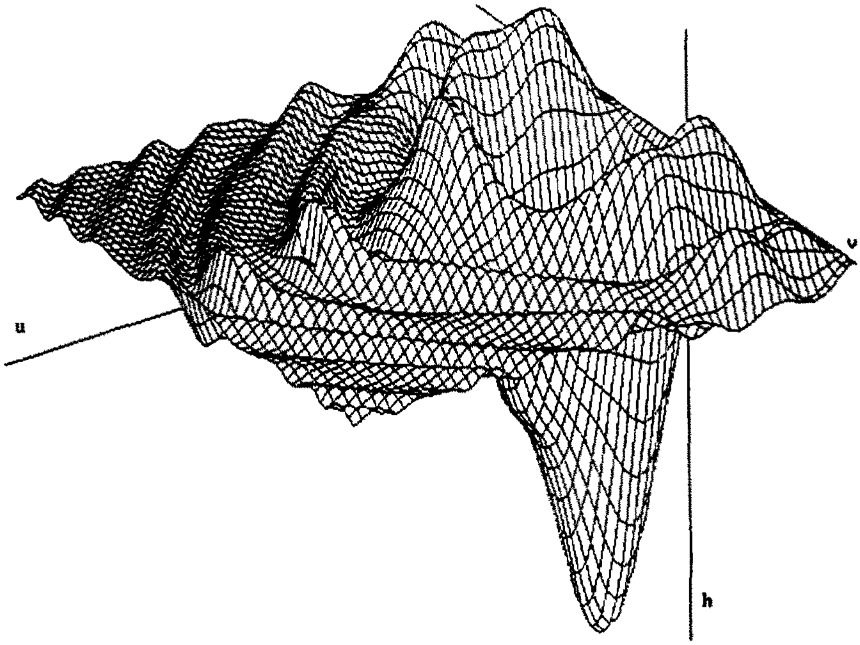

Fig. 3. Green's function of a semi-infinite one-sided clamped orthotropic plate for $s = ct/\xi^2 = 0.2$ in the domain $0 \leqslant u = x/\xi \leqslant 5$, $-3 \leqslant v = y/\xi \leqslant 3$. Maximum deflection $h_{max} = 0.170$.

of the deformed plate at some specific physical times are given in Figs 3, 4, 5, whereas Figs 6 and 7 show families of deformation curves for different times. Therein we use the dimensionless variables $u = x/\xi$, $v = y/\xi$, $s = ct/\xi^2$, $h = H_\xi/c$.

3.3. *The fundamental solution*

By shifting the boundary to infinity in the same way as in Section 2.4 we easily deduce, from the formula in Section 3.2, a fundamental solution E of the operator $\partial^4/\partial x^4 + \partial^4/\partial y^4 + c^{-2} \cdot \partial^2/\partial t^2$:

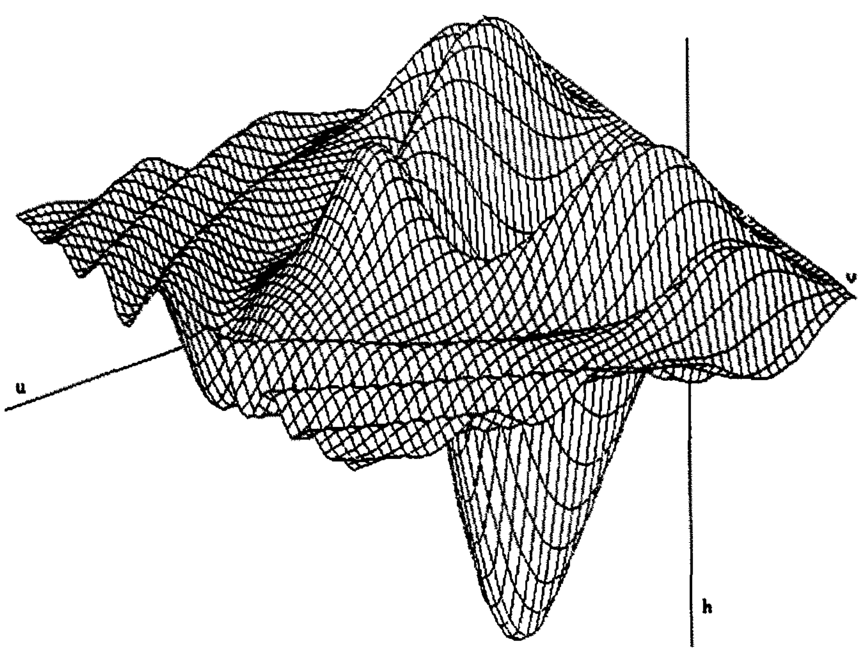

Fig. 4. Green's function of a semi-infinite one-sided clamped orthotropic plate for $s = ct/\xi^2 = 0.5$ in the domain $0 \leqslant u = x/\xi \leqslant 5$, $-5 \leqslant v = y/\xi \leqslant 5$. Maximum deflection $h_{max} = 0.139$.

$$E = \frac{c|y|Y(t)}{16\sqrt{\pi}} \int_0^{ct} \left\{ J_{-1/4}\left(\frac{y^2}{8\sqrt{c^2t^2-\tau^2}}\right)^2 - J_{1/4}\left(\frac{y^2}{8\sqrt{c^2t^2-\tau^2}}\right)^2 \right\} \cdot \sin\left(\frac{x^2}{4\tau} + \frac{\pi}{4}\right) \frac{d\tau}{\sqrt{\tau}\sqrt{c^2t^2-\tau^2}}.$$

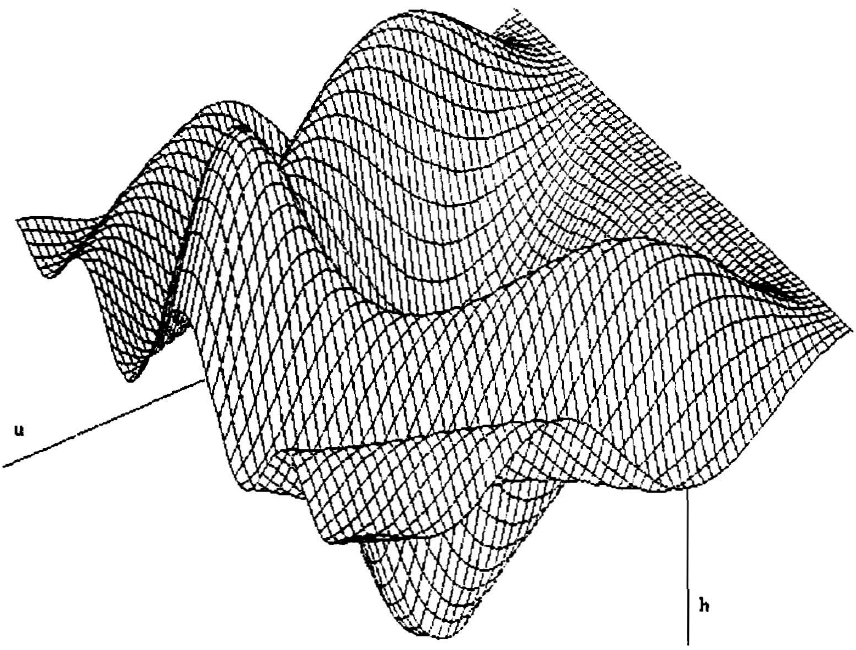

Fig. 5. Green's function of a semi-infinite one-sided clamped orthotropic plate for $s = ct/\xi^2 = 1.0$ in the domain $0 \leqslant u = x/\xi \leqslant 5$, $-5 \leqslant v = y/\xi \leqslant 5$. Maximum deflection $h_{max} = 0.093$.

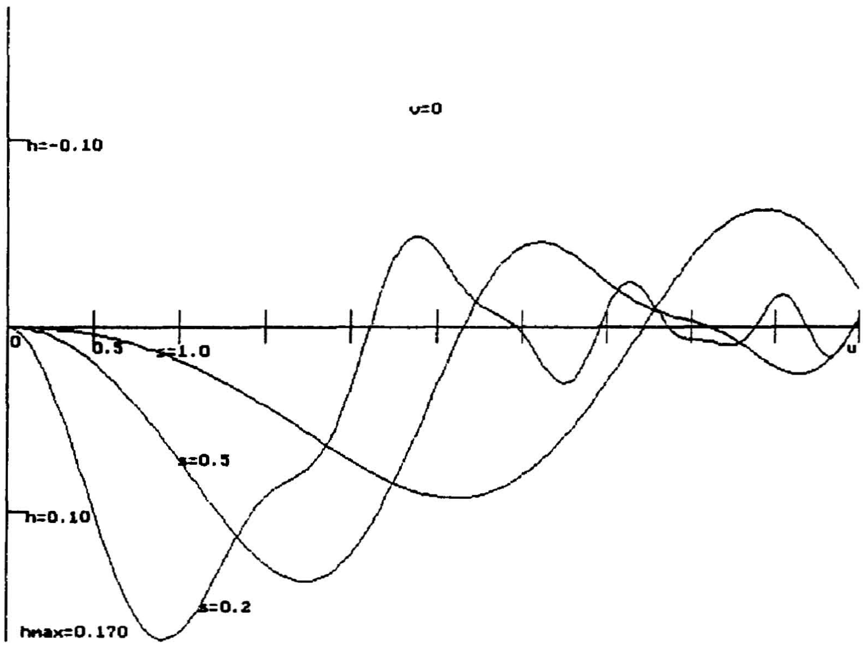

Fig. 6. Green's function of a semi-infinite one-sided clamped orthotropic plate: deflection of the midline $y = 0$ at the times $s = ct/\xi^2 = 0.2, 0.5, 1.0$.

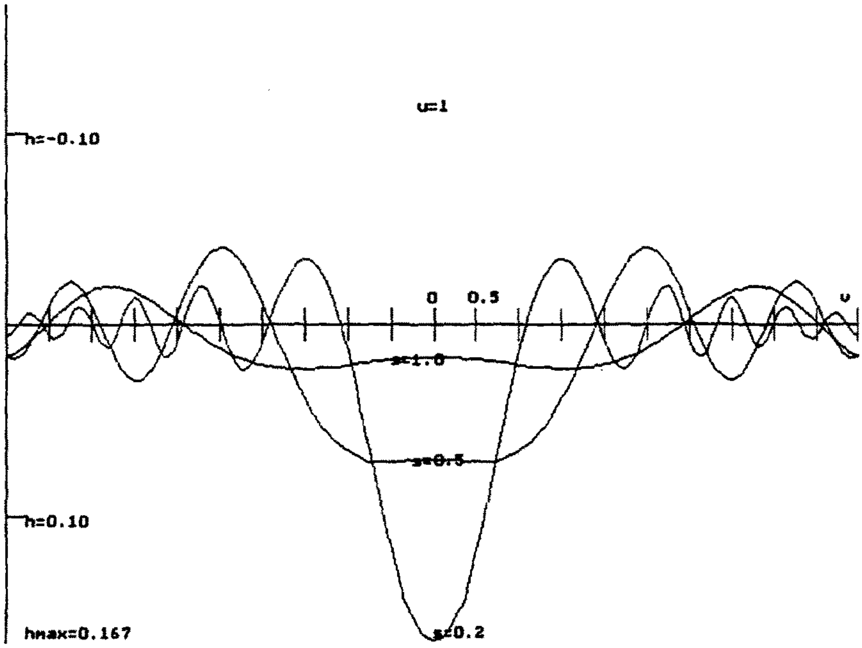

Fig. 7. Green's function of a semi-infinite one-sided clamped orthotropic plate: deflection of the line $x = \xi$ at the times $s = ct/\xi^2 = 0.2, 0.5, 1.0$.

E is the unique fundamental solution of the above operator satisfying the condition

$$\exists\alpha > 0 : e^{-\alpha t} E \in \mathcal{S}'$$

(Ortner, 1989). Let us remark that E can be given a somewhat more symmetric appearance with respect to x and y if the Fourier transform is applied to both spatial variables x, y, thus starting from the fundamental solution $(cY(t)/\alpha)\sin(\alpha ct)$ of the operator $\alpha^2 + c^{-2}\cdot(d^2/dt^2)$. One derives:

$$E = \frac{cY(t)}{4\pi}\int_0^\infty [1 - C(a)^2 - S(a)^2 - C(b)^2 - S(b)^2]\frac{d\tau}{\sqrt{1+\tau^4}},$$

where C and S denote the Fresnel integrals and

$$a = \frac{(x+\tau y)^2}{4ct\sqrt{1+\tau^4}}, \quad b = \frac{(x-\tau y)^2}{4ct\sqrt{1+\tau^4}}.$$

4. THE TRANSVERSE VIBRATIONS OF A SEMI-INFINITE, ISOTROPIC PLATE CLAMPED ALONG ITS BOUNDARY

4.1. *Statement of the problem*

Instead of the special orthotropic plate which corresponds to eqn (2) in Section 3, we consider here the deformation u of an isotropic plate. It is governed by the equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{1}{c^2}\cdot\frac{\partial^2 w}{\partial t^2} = \Delta_2^2 w + \frac{1}{c^2}\cdot\frac{\partial^2 w}{\partial t^2} = \frac{p}{K} \tag{3}$$

[cf. Nowacki, 1974, p. 291, eqn (31)]. We use the same coordinate system and prescribe the same boundary conditions as in Section 3. $F = F_\xi(x, y, t)$ denotes the Green's function, i.e. F solves eqn (3) with $p/K = \delta(x-\xi)\otimes\delta(y)\otimes\delta(t)$, $F(0, y, t) = (\partial F/\partial x)(0, y, t) = 0$, $F = 0$ for $t < 0$.

4.2. *Application of the Fourier–Laplace transform*

Upon applying, to F_ξ, the Fourier transform with respect to y and thereafter the Laplace transform with respect to t, we encounter a similar situation as in Section 2.2: $f_\xi := \mathscr{L}\mathscr{F}F_\xi$ is the Green's function of the ordinary differential operator $(d^4/dx^4) - 2\eta^2(d^2/dx^2) + (p^2/c^2) + \eta^4$ on the interval $(0, \infty)$. Exploiting the fact that $f_\xi(x, y, p)$ is real and its symmetry as regards x, ξ we can assume f_ξ to be of the following form:

$$f_\xi = \operatorname{Re}(a\,e^{-\lambda|x-\xi|} + b\,e^{-\lambda(x+\xi)}) + d\operatorname{Re}(e^{-\lambda x - \bar{\lambda}\xi}), \quad \lambda = \sqrt{\eta^2 + ip/c},\ \operatorname{Re}\lambda > 0,$$

with $a, b \in \mathbb{C}$, $d \in \mathbb{R}$. By the clamping conditions in $x = 0$ and by the continuity and jump conditions in $x = \xi$ (see Section 2.2), a, b, d are determined as $a = ic/2p\lambda$, $d = -(c^2/p^2)\operatorname{Re}\lambda$, $b = -a-d$. This yields:

$$f_\xi = \operatorname{Re}\left(\frac{ic}{2p\lambda}e^{-\lambda|x-\xi|}\right) - \operatorname{Re}\left(\frac{ic}{2p\lambda}e^{-\lambda(x+\xi)}\right) - \frac{c^2}{p^2}\operatorname{Re}\lambda \cdot \operatorname{Re}(e^{-\lambda x - \bar{\lambda}\xi} - e^{-\lambda(x+\xi)}).$$

In the first two terms of this sum, the Fourier–Laplace transformation can easily be inverted. With the help of some known formulae (see Oberhettinger, 1957, p. 13; Oberhettinger and Badii, 1973, p. 338), we find:

$$\begin{aligned}\mathscr{L}^{-1}\mathscr{F}^{-1}\left(\frac{1}{p\lambda}e^{-\lambda|x-\xi|}\right) &= \frac{1}{\pi}\mathscr{L}^{-1}\left(\frac{1}{p}\int_0^\infty \frac{\cos(y\eta)}{\lambda}e^{-\lambda|x-\xi|}\,d\eta\right)\\ &= \frac{1}{\pi}\mathscr{L}^{-1}\left(\frac{1}{p}K_0(\sqrt{ip/c}\cdot\sqrt{(x-\xi)^2+y^2})\right)\\ &= \frac{Y(t)}{2\pi}\int_0^t \exp\left(-i\frac{(x-\xi)^2+y^2}{4c\tau}\right)\frac{d\tau}{\tau}\end{aligned}$$

and hence

$$F_\xi = \frac{cY(t)}{4\pi}\int_0^t\left[\sin\left(\frac{(x-\xi)^2+y^2}{4c\tau}\right) - \sin\left(\frac{(x+\xi)^2+y^2}{4c\tau}\right)\right]\frac{d\tau}{\tau} + Z$$

with

$$Z = \mathscr{L}^{-1}\mathscr{F}^{-1}\left(-\frac{c^2}{p^2}\operatorname{Re}\lambda\cdot\operatorname{Re}(e^{-\lambda x-\bar{\lambda}\xi} - e^{-\lambda(x+\xi)})\right).$$

$F_\xi - Z$ coincides with the Green's function of a vibrating isotropic plate which is simply supported at the boundary, i.e. $w(0, y, t) = (\partial^2 w/\partial x^2)(0, y, t) = 0$. In this case, the Green's function can be deduced immediately from the fundamental solution E of the differential operator, i.e.

$$E = \frac{cY(t)}{4\pi}\int_0^t \sin\left(\frac{x^2+y^2}{4c\tau}\right)\frac{d\tau}{\tau} = \frac{cY(t)}{8}\left[1 - \frac{2}{\pi}Si\left(\frac{x^2+y^2}{4ct}\right)\right]$$

(cf. Ortner, 1980, p. 163) by the method of image points. In the following, we aim at giving an integral representation for Z.

4.3. *Integral representation for the "clamping term" Z*

$$Z=\mathscr{L}^{-1}\mathscr{F}^{-1}\left(\frac{c^2}{4p^2}(\lambda+\bar{\lambda})\cdot(\mathrm{e}^{-\lambda x}-\mathrm{e}^{-\bar{\lambda}x})\cdot(\mathrm{e}^{-\lambda\xi}-\mathrm{e}^{-\bar{\lambda}\xi})\right)=A+B+C,$$

where

$$A=\mathscr{L}^{-1}\mathscr{F}^{-1}\left(\frac{c^2}{2p^2}\operatorname{Re}(\lambda\,\mathrm{e}^{-\lambda(x+\xi)})\right),\quad B=(\partial_x+\partial_\xi)\mathscr{L}^{-1}\mathscr{F}^{-1}\left(\frac{c^2}{2p^2}\operatorname{Re}(\mathrm{e}^{-\lambda x-\bar{\lambda}\xi})\right),$$

$$C=\mathscr{L}^{-1}\mathscr{F}^{-1}\left(\frac{c^2}{2p^2}\operatorname{Re}(\lambda\,\mathrm{e}^{-\bar{\lambda}(x+\xi)})\right).$$

To begin with, let us treat A, which is the easiest of these terms;

$$\begin{aligned}\partial_t A&=\mathscr{L}^{-1}\mathscr{F}^{-1}\left(\frac{c^2}{2p}\operatorname{Re}(\lambda\,\mathrm{e}^{-\lambda(x+\xi)})\right),\\&=\frac{c^2}{2}\operatorname{Re}\left(\partial_x^2\mathscr{L}^{-1}\mathscr{F}^{-1}\left(\frac{1}{p\lambda}\mathrm{e}^{-\lambda(x+\xi)}\right)\right),\\&=\frac{c^2}{2}\operatorname{Re}\left(\partial_x^2\frac{Y(t)}{2\pi}\int_0^t\exp\left(-i\frac{(x+\xi)^2+y^2}{4c\tau}\right)\frac{\mathrm{d}\tau}{\tau}\right),\end{aligned}$$

using the formulae given in Section 4.2. Because

$$\frac{\partial}{\partial\alpha}\int_0^t\mathrm{e}^{-i\alpha/\tau}\frac{\mathrm{d}\tau}{\tau}=-\frac{\mathrm{e}^{-i\alpha/t}}{\alpha},$$

we obtain:

$$\partial_t A=-\partial_x\left(\frac{c^2(x+\xi)Y(t)}{2\pi z^2}\cos\left(\frac{z^2}{4ct}\right)\right)\quad\text{where } z^2:=(x+\xi)^2+y^2.$$

Now a partial integration yields:

$$\begin{aligned}A&=-\frac{c^2Y(t)}{2\pi}\partial_x\left(\frac{(x+\xi)t}{z^2}\cos\left(\frac{z^2}{4ct}\right)-\frac{x+\xi}{4c}\int_0^t\sin\left(\frac{z^2}{4c\tau}\right)\frac{\mathrm{d}\tau}{\tau}\right)\\&=\frac{c^2tY(t)}{2\pi}\cdot\frac{(x+\xi)^2-y^2}{z^4}\cos\left(\frac{z^2}{4ct}\right)+\frac{cY(t)}{8\pi}\int_0^t\sin\left(\frac{z^2}{4c\tau}\right)\frac{\mathrm{d}\tau}{\tau}.\end{aligned}$$

In order to represent B as explicitly as possible, we make use of the convolution exchange theorem (with respect to the Fourier and the Laplace transform):

$$B=\frac{c^2}{2}(\partial_x+\partial_\xi)\operatorname{Re}\left(\mathscr{L}^{-1}\mathscr{F}^{-1}\left(\frac{\mathrm{e}^{-\lambda x}}{p}\right)*\mathscr{L}^{-1}\mathscr{F}^{-1}\left(\frac{\mathrm{e}^{-\bar{\lambda}\xi}}{p}\right)\right).$$

The carrying out of the inverse transformations as in Section 4.2 yields:

$$B = \frac{c^2 Y(t)}{2\pi^2}(\partial_x+\partial_\xi)\left\{x\xi \operatorname{Re}\left[r^{-2}\exp\left(-\frac{ir^2}{4ct}\right)*\rho^{-2}\exp\left(\frac{i\rho^2}{4ct}\right)\right]\right\},$$

where $r^2 = x^2+y^2$, $\rho^2 = \xi^2+y^2$. When writing out the convolution with respect to y, t, we obtain:

$$B = \frac{c^2 Y(t)}{2\pi^2}(\partial_x+\partial_\xi)\left\{x\xi \int_{-\infty}^{\infty} \frac{d\eta}{(x^2+(y-\eta)^2)(\xi^2+\eta^2)} \cdot\int_0^t \cos\left(\frac{x^2+(y-\eta)^2}{4c\tau}-\frac{\xi^2+\eta^2}{4c(t-\tau)}\right)d\tau\right\}.$$

The determination of C takes a similar course to that of B. The convolution exchange theorem yields:

$$C = -\frac{c^2(x+\xi)Y(t)}{2\pi^2}\operatorname{Re}\left[\exp\left(-\frac{iy^2}{4ct}\right)\cdot \operatorname{Pf}\frac{1}{y^2}*z^{-2}\exp\left(\frac{iz^2}{4ct}\right)\right].$$

Here, $z^2 = (x+\xi)^2+y^2$ as above, and Pf denotes the finite part in the sense of Schwartz (1966, p. 38). Writing out the convolution with respect to y and t furnishes:

$$C = -\frac{c^2(x+\xi)Y(t)}{2\pi^2}\left\langle \operatorname{Pf}\frac{1}{\eta^2}, f(\eta)\right\rangle, \quad \text{with } f(\eta) = \int_0^t \cos\left(\frac{\eta^2}{4c\tau}-\frac{q^2}{4c(t-\tau)}\right)\frac{d\tau}{q^2},$$

where $q^2 = (x+\xi)^2+(y-\eta)^2$, and the brackets denote evaluation in the sense of the theory of distributions. This means

$$\left\langle \operatorname{Pf}\frac{1}{\eta^2}, f(\eta)\right\rangle = \int_0^\infty \frac{f(\eta)+f(-\eta)-2f(0)}{\eta^2}d\eta,$$

and hence

$$C = -\frac{c^2(x+\xi)Y(t)}{\pi^2}\int_0^\infty \frac{d\eta}{\eta^2}\int_0^t \left[\frac{1}{q^2}\cos\left(\frac{\eta^2}{4c\tau}-\frac{q^2}{4c(t-\tau)}\right)-\frac{1}{z^2}\cos\left(\frac{z^2}{4c(t-\tau)}\right)\right]d\tau.$$

These double integrals have not yet been evaluated numerically.

4.4. *Clamped semi-infinite elastically supported plate*

The considerations in Section 4.2 also allow us to represent the Green's function of the Dirichlet problem for the operator $\Delta_2^2+k^2$ in the half-plane, which describes the static deformation of a clamped semi-infinite elastically supported plate due to a concentrated force. With the abbreviations of Section 4.2, the function $(\mathscr{F}^{-1}f_\xi)(x,y) =: L_\xi(x,y)$ is the solution of this problem i.e.

$$\Delta_2^2 L_\xi+k^2 L_\xi = \delta(x-\xi)\otimes\delta(y), \quad x\geqslant 0, \quad y\in\mathbb{R}, \quad L_\xi(0,y)=0, \quad \frac{\partial L_\xi}{\partial x}(0,y)=0.$$

Putting $k = p/c$ we obtain

$$L_\xi(x,y) = -\frac{1}{2\pi k}[\text{kei}\,(\sqrt{k((x-\xi)^2+y^2)})-\text{kei}\,(\sqrt{k((x+\xi)^2+y^2)})]+\mathscr{L}Z.$$

The first two terms constitute the Green's function of an elastically supported plate which is simply supported along the boundary. In this case, the Green's function can be deduced from the fundamental solution, which is given in Ortner (1980, p. 159) by the method of images. For the clamping term, we have:

$$\mathscr{L}Z = -\frac{2}{k^2}\mathscr{F}^{-1}(\text{Re}\,\lambda\,\text{Im}\,e^{-\lambda x}\,\text{Im}\,e^{-\lambda\xi})$$

$$= -\frac{2}{\pi k^2}\int_0^\infty \kappa_1\,e^{-\kappa_1(x+\xi)}\sin(\kappa_2 x)\sin(\kappa_2\xi)\cos(\eta y)\,d\eta,$$

where

$$\kappa_1 = \sqrt{\frac{1}{2}(\eta^2+\sqrt{\eta^4+k^2})},\quad \kappa_2 = \sqrt{\frac{1}{2}(-\eta^2+\sqrt{\eta^4+k^2})}.$$

REFERENCES

Colombo, S. and Lavoine, J. (1972). *Transformations de Laplace et de Mellin.* Gauthier-Villars, Paris.

Erdélyi, A. (1956). *Asymptotic Expansions.* Dover, New York.

Girkmann, K. (1963). *Flächentragwerke.* Springer, Wien.

Nowacki, W. (1974). *Baudynamik.* Springer, Wien.

Oberhettinger, F. (1957). *Tabellen zur Fourier Transformation.* Springer, Berlin.

Oberhettinger, F. and Badii, L. (1973). *Tables of Laplace Transforms.* Springer, Berlin.

Ortner, N. (1978). The initial-boundary value problem of the transverse vibrations of a homogeneous beam clamped at one end and infinitely long in one direction. *Bull. Polish Acad. Sci. Math.* **26,** 389.

Ortner, N. (1980). Regularisierte Faltung von Distributionen. *Z. Angew. Math. Phys.* **31,** 133.

Ortner, N. (1982). Problem 616. *Nieuw Archief Wisk.* **30**(3), 215.

Ortner, N. (1989). Construction of fundamental solutions, to appear in *Topics in Boundary Element Research* (Edited by C. A. Brebbia). (In press.)

Parkus, H. (1966). *Mechanik der festen Körper,* 2nd Edition. Springer, Wien.

Schwartz, L. (1966). *Théorie des distributions,* Nouv. éd. Hermann, Paris.

Todhunter, I. and Pearson, K. (1893). *A History of the Theory of Elasticity and of the Strength of Materials,* Vol. 2, part II. The University Press, Cambridge, U.K.